QUASIANALYTICITY AND PLURIPOLARITY

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Abstract. We show that the graph

$$\Gamma_f = \{(z, f(z)) \in \mathbb{C}^2 : z \in S\}$$

in $\mathbb{C}^2$ of a function f on the unit circle S which is either continuous and quasianalytic in the sense of Bernstein or C^∞ and quasianalytic in the sense of Denjoy is pluripolar.

1. Introduction

A set K in $\mathbb{C}^n$ is called *pluripolar* if there is a plurisubharmonic (psh) function $u \not\equiv -\infty$ which is equal to $-\infty$ on K. As an example, if K is an analytic set given by an equation $h = 0$, where h is a holomorphic function, then $u = \log |h|$ is psh and equal to $-\infty$ on K.

Let f be a function on the unit circle S and let

$$\Gamma_f = \{(z, f(z)) \in \mathbb{C}^2 : z \in S\}$$

be the graph of f in $\mathbb{C}^2$.

The set Γ_f is always pluripolar when f is a *real-analytic* function. In [DF], Diederich and Fornæss give an example of a C^∞ function f with non-pluripolar graph in $\mathbb{C}^2$. The paper [LMP] contains an example of a holomorphic function f on the unit disk U, continuous up to the boundary, such that the graph of f over S is not pluripolar as a subset of $\mathbb{C}^2$. Thus *a priori* the pluripolarity of graphs of functions on S is indeterminate.

In this paper we prove that graphs of *quasianalytic* functions are still pluripolar (all necessary definitions can be found in the next section). More precisely,

Theorem 1.1. *If $f : S \to \mathbb{C}$ is quasianalytic in the sense of Bernstein or Denjoy, then the set $\Gamma_f \subset \mathbb{C}^2$ is pluripolar.*

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We also prove

Theorem 1.2. *Let $f : S \to \mathbb{C}$ be a C^∞ function. If f belongs to a Gevrey class $G\{L_j\}$, where $L_j \geq j$ satisfies $L_j = o(j^2)$ as $j \to \infty$, then the set Γ_f is pluripolar.*

The function f in the example of [DF] is in a Gevrey class $G\{L_j\}$ with $L_j = C^j$; thus, not all functions in a Gevrey class have pluripolar graphs. On the other hand, taking $L_j = j^{3/2}$, we see that there are functions in a Gevrey class that always have pluripolar graphs but which are *not* quasianalytic; i.e., in the C^∞ category, "quasianalytic" implies "pluripolar graph" but not conversely.

Since any continuous function is the difference of two continuous functions which are quasianalytic in the sense of Bernstein, we arrive at a surprising result.

Corollary 1.3. *Any continuous function defined on S is the difference of two continuous functions with pluripolar graphs.*

According to [M], any C^∞ function on the unit circle is the difference of two C^∞ functions which are quasianalytic in the sense of Denjoy. Hence we have the C^∞ analogue of Corollary 1.3.

Corollary 1.4. *Any C^∞ function on the unit circle is the difference of two C^∞ functions with pluripolar graphs.*

In the next section we recall all necessary facts and definitions. In Section 3 we prove that quasianalytic functions in the sense of Bernstein have *negligible* and, consequently, pluripolar graphs. To deal with C^∞ functions we establish in Section 4 a criterion (Theorem 4.2) for pluripolarity: if the total Monge–Ampère masses of a sequence of multipole pluricomplex Green functions are uniformly bounded from above, then the set where the limit of this sequence is equal to $-\infty$ is pluripolar. This criterion implies Corollary 4.4 which allows us to verify the pluripolarity of a set by constructing sequences of holomorphic mappings on annuli with appropriate radii. Theorem 1.1 for quasianalytic functions in the sense of Denjoy and Theorem 1.2 are proved in Section 5. For this we interpolate f at the n-th roots of unity and an arbitrary point $z_0 \in S$ by Lagrange trigonometric polynomials L_n and show that these interpolants are uniformly bounded on annuli $A(t_n) = \{1/t_n < |z| < t_n\}$, where

$$\limsup_{n\to\infty} \sqrt{n}\, \log t_n = \infty.$$

This formula is exactly what we need to apply Corollary 4.4.

We would like to thank Al Taylor for introducing us to the notion of quasianalytic functions.

2. Basic definitions and facts

For a continuous function f on S, we consider the approximation numbers $E_n(f) = \inf \|f - p_n\|_S$, where p_n runs over all trigonometric polynomials of degree at most n, i.e.,

$$p_n(z) = \sum_{k=-n}^{n} c_k z^k,\ z \in S,$$

and $\|\cdot\|_S$ is the uniform norm. A continuous function $f : S \to \mathbb{C}$ is called *quasianalytic in the sense of Bernstein* if

$$\liminf_{n\to\infty} E_n^{1/n}(f) < 1.$$

This class contains some continuous functions which are nowhere differentiable. We refer to [T] for further details.

Let $f : S \to \mathbb{C}$ be a C^∞ function with Fourier series given by

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k,\ z = e^{i\theta}. \tag{1}$$

The L^2 norm $M_j(f)$ of the j-th derivative $\widetilde{f}^{(j)}$, where $\widetilde{f}(\theta) = f(e^{i\theta})$, is given by

$$M_j^2(f) = \frac{1}{2\pi}\int_0^{2\pi} |\widetilde{f}^{(j)}(\theta)|^2 d\theta = \sum_{k=-\infty}^{\infty} k^{2j}|c_k|^2. \tag{2}$$

The sequence $\{M_j(f)\}$ is increasing and logarithmically convex. Following [Ka], we consider the associated function $\tau_f(r)$ (associated to the sequence $\{M_j(f)\}$) defined by

$$\tau_f(r) = \inf_{j\geq 0} \frac{M_j(f)}{r^j}\ ,\ r > 0.$$

We can assume that $M_j(f)$ increases faster than R^j for any $R > 0$, or else f is a trigonometric polynomial. Then $\tau_f(r) > 0$ is a decreasing function with $\lim_{r\to\infty} \tau_f(r) = 0$. The function $-\log \tau_f(r)$ is a convex, increasing function of $\log r$.

Now we recall some facts about quasianalytic and Gevrey classes of smooth functions (see [Ka] and [KP]). Given an increasing sequence $\{M_j\}$ which is logarithmically convex, the class $C^\#\{M_j\}$ consists of all smooth functions $f : S \to \mathbb{C}$ satisfying the estimate $M_j(f) \leq R^j M_j$ for all j with a constant R depending on f. The class $C^\#\{M_j\}$ is called quasianalytic if every function in $C^\#\{M_j\}$ which vanishes to infinite order at some point in S must be identically equal to 0. Let $\tau(r) = \inf_{j\geq 0}(M_j/r^j)$ be the associated function to the sequence M_j.

The Denjoy-Carleman theorem states that the class $C^\#\{M_j\}$ is quasianalytic if and only if

$$\int_1^\infty \frac{\log \tau(r)}{1+r^2}\, dr = -\infty. \tag{3}$$

A smooth function f is called *quasianalytic in the sense of Denjoy* if the class $C^\#\{M_j(f)\}$ is quasianalytic.

Let $\{L_j\}$ be a sequence such that $j \le L_j \le CL_{j-1}$ holds for all $j > 0$ with some constant $C > 0$ independent of j. The *Gevrey class* $G\{L_j\}$ consists of all smooth functions f which satisfy $M_j(f) \le (C'L_j)^j$ for all j, where the constant $C' > 0$ depends on f. The Gevrey class $G\{L_j\}$ is quasianalytic if and only if $\sum_{j=0}^\infty (1/L_j) = \infty$.

3. Quasianalytic functions in the sense of Bernstein

We will consider a trigonometric polynomial $p(e^{i\theta}) = \sum_{k=-n}^n c_k e^{ik\theta}$ as the restriction to S of the rational function $p(z) = \sum_{k=-n}^n c_k z^k$. The following lemma is a simple version of a Bernstein–Walsh inequality for the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Lemma 3.1. *Let p be a trigonometric polynomial on S of degree n. Then $|p(z)| \le \|p\|_S e^{nV(z)}$ for all $z \in \mathbb{C}^*$ where $V(z) = |\log |z||$.*

Proof. Assume that $\|p\|_S = 1$. The function $u(z) = (\log |p(z)|)/n$ is subharmonic when $z \neq 0$. Clearly $v(z) = u(z) - V(z)$ is subharmonic on $U \setminus \{0\}$ and on $\mathbb{C} \setminus \overline{U}$. Also v is bounded near zero and near infinity. Since $v \le 0$ on S, we see that $v \le 0$ when $z \neq 0$. Thus $|p(z)| \le e^{nV(z)}$ and the lemma is proved. □

Now we can prove our first result regarding the pluripolarity of graphs.

Theorem 3.2. *If a function $f : S \to \mathbb{C}$ is quasianalytic in the sense of Bernstein, then Γ_f is pluripolar in $\mathbb{C}^2$.*

Proof. We can find $c < 1$, a sequence of positive integers $\{n_k\}$, and a corresponding sequence of trigonometric polynomials $\{p_{n_k}\}$ with

$$E_{n_k}(f) = \|f - p_{n_k}\|_S \le c^{n_k}.$$

Without loss of generality, we may assume $\|f\|_S \le 1/2$ so that $\|p_{n_k}\|_S \le 1$; by Lemma 3.1 $|p_{n_k}(z)| \le e^{n_k V(z)}$ for each k and for all $z \in \mathbb{C}^*$.

Define functions $v_k(z,w) := \frac{1}{n_k} \log |w - p_{n_k}(z)|$ on $\mathbb{C}^* \times \mathbb{C}$. Then

$$v_k(z,w) \le \max\left\{V(z), \frac{1}{n_k}\log|w|\right\} + \frac{\log 2}{n_k}.$$

Also, if $(z,w) \in \Gamma_f$, i.e., $|z| = 1$ and $w = f(z)$, then $v_k(z,w) \leq \log c$. Let $v(z,w) = \sup_k v_k(z,w)$. The function v is bounded above on compacta in $\mathbb{C}^* \times \mathbb{C}$. Moreover, $v \leq \log c$ on Γ_f and since $p_{n_k}(z) \to f(z)$ as $k \to \infty$, $v \geq 0$ on $(S \times \mathbb{C}) \setminus \Gamma_f$. Thus the function v is not upper semicontinuous on Γ_f, i.e.,

$$\Gamma_f \subset \{(z,w) \in \mathbb{C}^* \times \mathbb{C} : v(z,w) < v^*(z,w) = \limsup_{(z',w') \to (z,w)} v(z',w')\}$$

is a negligible set. By [BT], negligible sets are pluripolar and our theorem is proved. □

4. A criterion for pluripolarity

To avoid notational confusion, we temporarily utilize $\mathbf{z}$ for a point $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$ and write $\|\mathbf{z}\|^2 = |z_1|^2 + \cdots + |z_n|^2$. Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ with a strongly psh defining function $\rho \in C^2(\overline{D})$. In [S] Sadullaev proved that for any set $E \subset D_r = \{\mathbf{z} \in D : \rho(\mathbf{z}) \leq r\}$, $r < 0$, there are positive constants $\alpha(r)$ and $\beta(r)$ depending only on r such that

$$\alpha(r) C^*(E) \leq \int_D |\omega^*(\mathbf{z}, E, D)| (dd^c\rho)^n \leq \beta(r) \left[C^*(E)\right]^{1/n}. \tag{4}$$

Here $\omega^*(\mathbf{z}, E, D) = \limsup_{\zeta \to \mathbf{z}} \omega(\zeta, E, D)$ is the upper semicontinuous regularization of the relative extremal function

$$\omega(\zeta, E, D) = \sup\{u(\zeta) : u \text{ psh in } D,\ u \leq 0,\ u|_E \leq -1\}$$

of the set E in D, and $C^*(E)$ is the outer capacity of E relative to D:

$$C^*(E) = \inf_V \sup_u \int_V (dd^c u)^n,$$

where u runs over all psh functions u on D such that $-1 \leq u \leq 0$ and V is any open set in D containing E (for a discussion of the Monge-Ampère operator, $(dd^c(\cdot))^n$, we refer the reader to [BT]).

We also consider weighted multipole pluricomplex Green functions $G(\mathbf{z}) = G(\mathbf{z}; a; \alpha)$ where $a = \{a_1, ..., a_m\}$ are points in D and $\alpha = \{\alpha_1, ..., \alpha_m\}$ are positive numbers:

$$G(\mathbf{z}) = \sup\{u(\mathbf{z}) : u \text{ psh in } D,\ u \leq 0,$$
$$u(\mathbf{z}) - \alpha_j \log ||\mathbf{z} - a_j|| = O(1),\ \mathbf{z} \to a_j,\ j = 1, ..., m\}.$$

It is known that G is continuous and psh in D, $(dd^cG)^N = 0$ on $D \setminus \{a_1, ..., a_m\}$, $G = 0$ on ∂D, and $G(\mathbf{z}) - \alpha_j \log ||\mathbf{z} - a_j|| = O(1)$, $\mathbf{z} \to a_j$, $j = 1, ..., m$. We refer to [D] and [L] for further properties of these functions. We have the following lemma.

Lemma 4.1. *Suppose that $D = \{\rho < 0\}$ is a strongly pseudoconvex domain in $\mathbb{C}^n$ as above. If $G(\mathbf{z})$ is a multipole Green function with poles in the set D_r, $r < 0$, then there is a positive constant $b(r)$ depending only on r such that*

$$\int_D |G|(dd^c\rho)^n \le b(r) \left(\int_D (dd^c G)^n \right)^{1/n}.$$

Proof. Fix t sufficiently large so that $E_t = \{G \le -t\} \subset D_{r/2}$. Then $t\omega^*(\mathbf{z}, E_t, D) = G_t(\mathbf{z}) = \max\{G(\mathbf{z}), -t\}$ and from [D, Theorem 4.2]

$$\int_D (dd^c G)^n = \int_D (dd^c G_t)^n. \tag{5}$$

If E is relatively compact in D, then by [K, Prop. 4.7.2]

$$C^*(E) = \int_D (dd^c \omega^*(\mathbf{z}, E, D))^n.$$

Hence by (4) and (5)

$$\int_D |G_t|(dd^c\rho)^n = t \int_D |\omega^*(\mathbf{z}, E_t, D)|(dd^c\rho)^n \le$$

$$t\beta(r/2) \left(\int_D (dd^c \omega^*(\mathbf{z}, E_t, D))^n \right)^{1/n} = \beta(r/2) \left(\int_D (dd^c G)^n \right)^{1/n}.$$

By monotone convergence

$$\int_D |G|(dd^c\rho)^n = \lim_{t\to\infty} \int_D |G_t|(dd^c\rho)^n,$$

and the lemma is proved. □

Now we can state a criterion for pluripolarity.

Theorem 4.2. *Under the hypotheses of Lemma 4.1 suppose that $\{G_j\}$ is a sequence of multipole Green functions on D with poles in D_r, $r < 0$, and*

$$\sup_j \int_D (dd^c G_j)^n \le A < \infty.$$

For $t > 0$, $s < 0$ let

$$E_t = \{\mathbf{z} \in D_s : \limsup_{j\to\infty} G_j(\mathbf{z}) < -t\}.$$

Then for the outer capacity of E_t relative to D we have the estimate

$$C^*(E_t) \le \frac{b(r)}{t\alpha(s)} A^{1/n}$$

where $\alpha(s)$ and $b(r)$ are positive constants depending only on s and r. In particular, the set

$$E = \{\mathbf{z} \in D : \lim_{j\to\infty} G_j(\mathbf{z}) = -\infty\}$$

is pluripolar.

Proof. Define

$$E_t^k = \{\mathbf{z} \in E_t : G_j(\mathbf{z}) < -t \text{ for all } j \ge k\}.$$

Let ω^* be the relative extremal function of E_t^k in D. Then $G_j \le t\omega^*$ for all $j \ge k$. Therefore, by (4)

$$\int\limits_D |G_j|(dd^c\rho)^n \ge t \int\limits_D |\omega^*|(dd^c\rho)^n \ge t\alpha(s)C^*(E_t^k).$$

By Lemma 4.1

$$\int\limits_D |G_j|(dd^c\rho)^n \le b(r)A^{1/n}.$$

Thus

$$C^*(E_t^k) \le \frac{b(r)}{t\alpha(s)} A^{1/n}.$$

Since $E_t^k \subset E_t^{k+1}$ and $E_t = \cup_k E_t^k$, by [K, Corollary 4.7.11]

$$C^*(E_t) = \lim_{k\to\infty} C^*(E_t^k).$$

Hence we get the first statement of our lemma.

It follows from this statement that for every $s < 0$ the set $E \cap D_s$ is pluripolar. Hence E is also pluripolar. $\square$

We include the following lemma because we were not able to find it in the literature. We denote by $g_D(z,w)$ the (negative) Green function of a domain $D \subset \mathbb{C}$ with pole at $w \in D$.

Lemma 4.3. *Fix r, a with $r > a > 1$ and let $A = \{z \in \mathbb{C} : 1/r < |z| < r\}$ be an annulus. Let $g(z) = g_A(z,1)$ be the Green function of A with pole at $w = 1$. Then*

$$\sup_{|z|=1} g(z) \le c(a)\log r,$$

where $c(a) < 0$ depends only on a.

Proof. The mapping

$$w = f_1(z) = \exp\left(\log r\left(1 + i\frac{2}{\pi}z\right)\right)$$

is a universal holomorphic covering map of A by the strip $T = \{z \in \mathbb{C} :\ 0 < \mathbf{Im}\ z < \pi\}$. Since $f_1(z) = 1$ if and only if

$$z = z_k = \frac{k\pi^2}{\log r} + \frac{\pi}{2}i,\ \ k = 0, \pm 1, \pm 2, ...$$

we see that

$$g_A(f_1(z), 1) = \sum_{k=-\infty}^{\infty} g_T(z, z_k).$$

Note that if $I_1 = \{x + i\pi/2 :\ -\pi^2/\log r \leq x \leq 0\}$, then $f_1(I_1) = \{|z| = 1\}$.

The mapping

$$f_2(z) = \log\left(i\frac{1-z}{1+z}\right)$$

maps the unit disk U conformally onto the strip T. Since $f_2(z'_k) = z_k$ and $f_2(I_2) = I_1$, where

$$z'_k = \frac{1 - e^{k\pi^2/\log r}}{1 + e^{k\pi^2/\log r}}$$

and

$$I_2 = \left[0, \frac{1 - e^{-\pi^2/\log r}}{1 + e^{-\pi^2/\log r}}\right],$$

we see that to prove the lemma we need to estimate the supremum M of

$$u(x) = \sum_{k=-\infty}^{\infty} g_U(x, z'_k) = \sum_{k=-\infty}^{\infty} \log\left|\frac{x - z'_k}{1 - z'_k x}\right|$$

when x varies over I_2.

If $k \leq -1$, then the maximum of the function $\log|(x - z'_k)/(1 - z'_k x)|$ is achieved when $x = 0$. Thus

$$M \leq \sum_{k=-\infty}^{-1} \log\frac{1 - e^{k\pi^2/\log r}}{1 + e^{k\pi^2/\log r}} = \sum_{k=-\infty}^{-1} \log\left(1 - \frac{2e^{k\pi^2/\log r}}{1 + e^{k\pi^2/\log r}}\right).$$

Since $\log(1 - x) \leq -x$,

$$M \leq -\sum_{k=-\infty}^{-1} \frac{2e^{k\pi^2/\log r}}{1 + e^{k\pi^2/\log r}} \leq -\sum_{k=-\infty}^{-1} e^{k\pi^2/\log r} = -\frac{e^{-\pi^2/\log r}}{1 - e^{-\pi^2/\log r}}.$$

Since $1 - e^{-x} \leq x$, we finally obtain

$$M \leq -\frac{e^{-\pi^2/\log r}}{\pi^2} \log r \leq -\frac{e^{-\pi^2/\log a}}{\pi^2} \log r.$$

□

The corollary below allows us to verify the conditions of Theorem 4.2 by constructing special sequences of holomorphic mappings of appropriate annuli.

Corollary 4.4. *Let E be a set in a ball $B \subset \mathbb{C}^n$. Suppose that there exists a sequence of arrays of points $W_k = \{w_{1k}, \ldots, w_{m_k k}\}$ in B and a sequence of numbers r_k such that, for each $\mathbf{z} \in E$, there exists a sequence of holomorphic mappings f_k of annuli $A_k = \{\zeta \in \mathbb{C} : 1/r_k < |\zeta| < r_k\}$ into B satisfying:*

(1) $f_k(e^{2\pi ij/m_k}) = w_{jk}$, $1 \leq j \leq m_k$;
(2) *there exists* $\zeta_k \in S$ *with* $f_k(\zeta_k) = \mathbf{z}$;
(3)

$$\limsup_{k\to\infty} m_k^{1-1/n} \log r_k = \infty.$$

Then the set E is pluripolar.

Proof. By passing to a subsequence we may assume that

$$\lim_{k\to\infty} m_k^{1-1/n} \log r_k = \infty.$$

Let B' be the ball concentric with B and of twice the radius. We let $G_k(\mathbf{z})$ denote the multipole Green function on B' with poles at the points in W_k of weight $m_k^{-1/n}$. Then

$$\int_{B'} (dd^c G_k)^n \leq A < \infty.$$

For $\mathbf{z} \in E$ we let $u_k(\zeta) = G_k(f_k(\zeta))$ so that $u_k(\zeta_k) = G_k(\mathbf{z})$. The functions $u_k(\zeta)$ are negative and subharmonic and have poles of order $m_k^{-1/n}$ at the points $e^{2\pi ij/m_k}$, $1 \leq j \leq m_k$. Letting $g(t) = g_R(t, 1)$ denote the Green function of the annulus $R := \{t \in \mathbb{C} : r_k^{-m_k} < |t| < r_k^{m_k}\}$ with pole at $t = 1$, we have $u_k(\zeta) \leq m_k^{-1/n} g(\zeta^{m_k})$. Note that $r_k^{m_k} \to \infty$; thus for k sufficiently large, $r_k^{m_k} > 2$. By Lemma 4.3 with $a = 2$, for such k

$$G_k(\mathbf{z}) = u_k(\zeta_k) \leq c(2) m_k^{-1/n} \log r_k^{m_k} = c(2) m_k^{1-1/n} \log r_k,$$

where $c(2) < 0$. Thus

$$\lim_{k\to\infty} G_k(\mathbf{z}) = -\infty.$$

By Theorem 4.2 the set E is pluripolar. $\square$

5. Quasianalytic functions in the sense of Denjoy

The proofs of Theorem 1.1 for Denjoy quasianalytic functions and of Theorem 1.2 will follow from a slightly more general, albeit slightly technical-looking, result. Given a smooth (C^∞) function f on S with associated function $\tau_f(r)$, we define

$$\log t_n = \min\left\{-\frac{\log r^3\tau_f(r)}{r} \,:\, 1 \le r \le n\right\}. \tag{6}$$

Proposition 5.1. *Let $f : S \to \mathbb{C}$ be a C^∞ function such that*

$$\limsup_{n\to\infty} \sqrt{n}\,\log t_n = \infty. \tag{7}$$

Then the set Γ_f is pluripolar.

Proof. Let $f : S \to \mathbb{C}$ be a smooth function with Fourier expansion (1). By (2) we have

$$|c_k| \le M_j(f)/|k|^j,\ |k| \ge 1,\ j \ge 0. \tag{8}$$

The idea of the proof is to interpolate f by trigonometric polynomials $L_n(f, z_0; z)$ at the n-th roots of unity and at some other point z_0 with $|z_0| = 1$. Lemma 5.2 provides estimates for $L_n(f, z_0; z)$ when z lies in an annulus $A(t) = \{z : 1/t \le |z| \le t\}$. It follows from these estimates that we may apply Corollary 4.4 using the sequence of arrays of points $W_n = \{(z, f(z)) : z^n = 1\}$, the annuli $A(t_n)$ and our hypothesis (7).

We proceed to define the appropriate interpolating trigonometric polynomials $\{L_n(f, z_0; z)\}$; recall these are rational functions restricted to S. To begin with, we let

$$L_n(f; z) = \sum_{r=0}^{n-1} a_{n,r} z^r + \sum_{r=1}^{n} \frac{b_{n,r}}{z^r},$$

where

$$a_{n,r} = \sum_{j=0}^{\infty} c_{r+nj},\ b_{n,r} = \sum_{j=0}^{\infty} c_{-r-nj},$$

and then we define

$$L_n(f, z_0; z) = L_n(f; z) + \frac{z^n - 1}{z_0^n - 1}\,(f(z_0) - L_n(f; z_0)).$$

Here z_0 is any point on the unit circle with $z_0^n \ne 1$. Note that $a_{n,r}$, $b_{n,r}$ are well defined since $\sum |c_k| < \infty$ from (8).

If $z_1, \dots, z_n$ are the n-th roots of unity, then $z_l^{r+nj} = z_l^r$ and, therefore,

$$L_n(f, z_0; z_l) = f(z_l), \ l = 0, 1, \dots, n,$$

i.e., $L_n(f, z_0; z)$ interpolates f at $z_0, \dots, z_n$. If $z_0^n = 1$ we define $L_n(f, z_0; z) = L_n(f; z)$. We remark that C_f will denote a constant depending on f which may vary from line to line.

Lemma 5.2. *There exists a constant C_f depending only on f so that for every $n \geq 1$, every z_0 with $|z_0| = 1$, and every $t > 1$, we have*

$$|L_n(f, z_0; z)| \leq C_f \left(1 + \sum_{r=1}^{n} r \tau_f(r) t^r \right)$$

for all z with $1/t \leq |z| \leq t$.

Proof. If $s \geq 2$ we have, using (8),

$$|a_{n,r}| \leq \sum_{j=0}^{\infty} |c_{r+nj}| \leq M_s(f) \sum_{j=0}^{\infty} \frac{1}{(r+nj)^s} \leq 2 \frac{M_s(f)}{r^s} \ ,$$

and similarly $|b_{n,r}| \leq 2M_s(f)/r^s$. By the definition of $\tau_f(r)$ we clearly have

$$\tau_f(r) = \min \left\{ M_0(f), \frac{M_1(f)}{r}, \frac{M_2(f)}{r^2}, \inf_{s \geq 3} \frac{M_s(f)}{r^s} \right\} = \inf_{s \geq 3} \frac{M_s(f)}{r^s}$$

for all $r > r_0(f)$. We conclude that $|a_{n,r}|$, $|b_{n,r}|$ are bounded above by $C_f \tau_f(r)$. Therefore

$$|L_n(f; z)| \leq C_f \left(1 + 2 \sum_{r=1}^{n} \tau_f(r) t^r \right)$$

holds for $1/t \leq |z| \leq t$. This gives the desired bound when $z_0^n = 1$.

Next, if $z_0^n \neq 1$, we note that

$$|(z_0^{nj} - 1)/(z_0^n - 1)| = |z_0^{n(j-1)} + z_0^{n(j-2)} + \dots + 1| \leq j,$$

moreover,

$$f(z_0) = \sum_{j=-\infty}^{\infty} c_j z_0^j = \sum_{r=0}^{n-1} \sum_{j=0}^{\infty} c_{r+nj} z_0^{r+nj} + \sum_{r=1}^{n} \sum_{j=0}^{\infty} \frac{c_{-r-nj}}{z_0^{r+nj}},$$

thus

$$\left|\frac{f(z_0)-L_n(f;z_0)}{z_0^n-1}\right| =$$

$$\left|\sum_{r=0}^{n-1}\sum_{j=0}^{\infty} c_{r+nj}\,\frac{z_0^{r+nj}-z_0^r}{z_0^n-1}+\sum_{r=1}^{n}\sum_{j=0}^{\infty}\frac{c_{-r-nj}(1-z_0^{nj})}{z_0^{r+nj}(z_0^n-1)}\right| \leq$$

$$\sum_{r=0}^{n-1}\sum_{j=0}^{\infty} j|c_{r+nj}|+\sum_{r=1}^{n}\sum_{j=0}^{\infty} j|c_{-r-nj}|.$$

For $s \geq 3$ we have, using (8),

$$\sum_{j=1}^{\infty} j|c_{\pm(r+nj)}| \leq \sum_{j=1}^{\infty}\frac{jM_s(f)}{(r+nj)^s} \leq \sum_{j=1}^{\infty}\frac{jM_s(f)}{n^s j^s} \leq 2\,\frac{M_s(f)}{n^s}\ .$$

Thus by the definition of $\tau_f(n)$,

$$\left|\frac{f(z_0)-L_n(f;z_0)}{z_0^n-1}\right| \leq 4n\tau_f(n).$$

Since $|z^n-1| \leq 2t^n$ for $|z| \leq t$, this estimate, together with the bound on $|L_n(f;z)|$, implies the lemma. $\square$

Using (6) we have defined the sequence $\{t_n\}$ via

$$t_n = \min\left\{\frac{1}{(r^3\tau_f(r))^{1/r}} : 1 \leq r \leq n\right\}.$$

Since the numbers t_n are decreasing, we obtain from (7) that $t_n > 1$ for each n; moreover $r^3\tau_f(r)t_n^r \leq 1$ for $r \leq n$. If $1/t_n \leq |z| \leq t_n$ it follows from the previous lemma that

$$|L_n(f,z_0;z)| \leq C_f\left(1+\sum_{r=1}^{n}\frac{r^3\tau_f(r)t_n^r}{r^2}\right) \leq C_f\left(1+\sum_{r=1}^{n}\frac{1}{r^2}\right) \leq 3C_f.$$

We conclude that for each n and for each z_0 with $|z_0| = 1$, the images of the annuli $A(t_n) = \{1/t_n \leq |z| \leq t_n\}$ under the mappings $h_n(z) = (z, L_n(f,z_0;z))$ are contained in a ball $B \subset \mathbb{C}^2$ centered at the origin and of radius R_f depending only on f. Thus by (7) and Corollary 4.4 the set Γ_f is pluripolar. $\square$

Before proceeding with the proofs of Theorems 1.1 (for functions quasianalytic in the sense of Denjoy) and 1.2 we make the following remarks. Note that the graph of f is pluripolar if and only if the graph of cf is pluripolar, where $c \neq 0$ is a constant. Multiplying f by a small

constant we may assume in the sequel that our smooth functions f verify $M_3(f) < 1/2$. Let

$$\widetilde{\tau}_f(r) := \inf_{s \geq 3} \frac{M_s(f)}{r^{s-3}} = \inf_{s \geq 0} \frac{M_{s+3}(f)}{r^s} < \frac{1}{2} \tag{9}$$

be the associated function for the shifted sequence $\{\tilde{M}_s\} = \{M_{s+3}(f)\}$; setting

$$\log \theta_f(n) = \min \left\{ -\frac{\log \widetilde{\tau}_f(r)}{r} \, : \, 1 \leq r \leq n \right\}, \tag{10}$$

it follows that

$$\log t_n \geq \log \theta_f(n) > 0.$$

From the definition of $\tau_f(r)$, as indicated in the proof of Lemma 5.2, we clearly have

$$r^3 \tau_f(r) = \min \left\{ M_0(f) r^3, M_1(f) r^2, M_2(f) r, \inf_{s \geq 3} \frac{M_s(f)}{r^{s-3}} \right\} = \widetilde{\tau}_f(r)$$

for all $r > r_0(f) \geq 0$. We show that if f is quasianalytic in the sense of Denjoy or if f belongs to a Gevrey class $G\{L_j\}$, where $L_j = o(j^2)$, then

$$\limsup_{n \to \infty} \sqrt{n} \log \theta_f(n) = \infty. \tag{11}$$

This implies condition (7) from Proposition 5.1 and finishes the proofs of Theorems 1.1 and 1.2.

We need the following lemma, whose proof we postpone until the end of this section.

Lemma 5.3. *Let $\tilde{h}(s) = h(e^s)$ be a positive, increasing, convex function of s on $[0, \infty)$ and let $\tilde{H}(x) = \min\{\tilde{h}(s) e^{-s} \, : \, 0 \leq s \leq x\}$. If $\tilde{H}(x) \leq C e^{-x/2}$ for all $x \geq 0$, then*

$$\int_0^\infty \tilde{h}(s) e^{-s} \, ds = \int_1^\infty \frac{h(t)}{t^2} dt < \infty.$$

We now prove Theorem 1.1 for quasianalytic functions in the sense of Denjoy.

Proof. The condition (3) for quasianalyticity holds for $\widetilde{\tau}_f(r)$ (equivalently, for $\{\tilde{M}_s\}$); in particular, since $r^3 \tau_f(r) = \widetilde{\tau}_f(r)$ for all $r > r_0(f)$, using (3) for $\tau_f(r)$, we have

$$\int_{r_0}^\infty \frac{\log \widetilde{\tau}_f(r)}{1 + r^2} \, dr = \int_{r_0}^\infty \frac{3 \log r}{1 + r^2} \, dr + \int_{r_0}^\infty \frac{\log \tau_f(r)}{1 + r^2} \, dr = -\infty;$$

i.e., letting $h(r) := -\log \widetilde{\tau}_f(r)$,

$$\int\limits_1^\infty \frac{h(t)}{1+t^2}dt = \infty. \tag{12}$$

We assume, for the sake of obtaining a contradiction, that (11) does not hold. Then, with $h(t) = -\log \widetilde{\tau}_f(t)$ and $\tilde{h}(s) = h(e^s)$, we have

$$H(t) = \log \theta_f(t) = \min\{h(r)/r : 1 \le r \le t\},$$

and it follows that there exists a constant $C > 0$ so that $H(n) < C/\sqrt{n}$ for every integer n. Since H is decreasing, $H(x) \le 2C/\sqrt{x}$ for all x. In terms of

$$\tilde{H}(x) = \min\{\tilde{h}(s)e^{-s} : 0 \le s \le x\} = H(e^x),$$
$$\tilde{H}(x) < 2Ce^{-x/2}, \ \forall\, x \ge 0.$$

By (9) we have $h(t) > 0$, so Lemma 5.3 implies that

$$\int_1^\infty \frac{h(t)}{t^2} < \infty, \tag{13}$$

which contradicts (12); i.e., the fact that $\widetilde{\tau}_f(r)$ (equivalently, $\{\tilde{M}_s\}$) defines a quasianalytic class. $\square$

We proceed with the proof of Theorem 1.2.

Proof. Writing $L_j = j^2/a_j$, $a_j \to \infty$, we see that $f \in G\{L_j\}$ satisfies

$$M_j(f)/r^j \le (Cj^2/(ra_j))^j$$

for all $j, r \ge 1$. Taking $j = [\sqrt{r}]$, the greatest integer in $\sqrt{r}$, we obtain

$$\widetilde{\tau}_f(r) \le r^3(C/a_{[\sqrt{r}]})^{[\sqrt{r}]};$$

hence

$$\frac{-\log \widetilde{\tau}_f(r)}{r} \ge -3\frac{\log r}{r} + \frac{[\sqrt{r}]}{r}\left(\log a_{[\sqrt{r}]} - \log C\right).$$

Since the nonnegative function $\log \theta_f(r)$ is decreasing, either $\log \theta_f(r) = c > 0$ for large r or else there is a sequence $r_k \to \infty$ such that

$$\log \theta_f(r_k) = -\frac{\log \widetilde{\tau}_f(r_k)}{r_k}.$$

In the first case the condition (11) is clearly satisfied. In the second case, if $n_k = [r_k] - 1$, then

$$\sqrt{n_k}\log \theta_f(n_k) \ge -3\frac{\sqrt{n_k}\log r_k}{r_k} + \frac{\sqrt{n_k}[\sqrt{r_k}]}{r_k}\left(\log a_{[\sqrt{r_k}]} - \log C\right) \to \infty.$$

$\square$

Proof of Lemma 5.3. Since the functions $\tilde{h}$ and $\tilde{H}$ are continuous, the set $E = \{x : \tilde{h}(x)e^{-x} = \tilde{H}(x)\}$ is closed. We have

$$\int_E \tilde{h}(x)e^{-x}\,dx = \int_E \tilde{H}(x)\,dx \le C\int_0^\infty e^{-x/2} < \infty.$$

Let $F = [0,\infty]\setminus E$. The set F is open and, therefore, $F = \cup(a_j, b_j)$, where $a_j, b_j \in E$ or $b_j = \infty$.

We first show that $\tilde{H}(x) = \tilde{h}(a_j)e^{-a_j}$ on $[a_j, b_j]$. Indeed, since $\tilde{H}(x)$ is decreasing, $\tilde{H}(x) \le \tilde{h}(a_j)e^{-a_j}$ on $[a_j, b_j]$. If $\tilde{h}(x)e^{-x} < \tilde{h}(a_j)e^{-a_j}$ for some $x \in [a_j, b_j]$, then let s be a point in $[a_j, x]$, where $\tilde{h}(x)e^{-x}$ attains its minimum. Clearly $s > a_j$. But then $\tilde{H}(s) = \tilde{h}(s)e^{-s}$ and $s \in E$. Thus $s = b_j$. Hence $\tilde{h}(x)e^{-x} \ge \tilde{h}(a_j)e^{-a_j}$ for all $x \in [a_j, b_j]$ and $\tilde{H}(x) = \tilde{h}(a_j)e^{-a_j}$.

Next we show that $b_j < \infty$ for all j. If not, then $\tilde{H}(x) = \tilde{h}(a_j)e^{-a_j} \le Ce^{-x/2}$ for all $x \ge a_j$ and hence $\tilde{h}(a_j) = 0$. This contradiction shows that $b_j < \infty$.

From the above results we have

$$\tilde{H}(x) = \tilde{h}(a_j)e^{-a_j} = \tilde{h}(b_j)e^{-b_j} \le Ce^{-b_j/2}$$

when $a_j \le x \le b_j$ Consequently, $\tilde{h}(a_j) \le Ce^{a_j - b_j/2}$ and $\tilde{h}(b_j) \le Ce^{b_j/2}$.

If $x = \alpha a_j + (1-\alpha)b_j$, $0 \le \alpha \le 1$, then

$$\tilde{h}(x) \le \alpha\tilde{h}(a_j) + (1-\alpha)\tilde{h}(b_j) \le Ce^{-b_j/2}\left(\alpha e^{a_j} + (1-\alpha)e^{b_j}\right)$$

and

$$\tilde{h}(x)e^{-x} \le Ce^{-b_j/2}\frac{\alpha e^{a_j} + (1-\alpha)e^{b_j}}{e^{\alpha a_j + (1-\alpha)b_j}} = Ce^{-b_j/2}\frac{\alpha e^{c_j} + 1 - \alpha}{e^{\alpha c_j}},$$

where $c_j = a_j - b_j$.

We split the intervals (a_j, b_j) into three separate types. The first type consists of all intervals having length at most 2; i.e., $-c_j \le 2$. On these intervals

$$\tilde{h}(x)e^{-x} \le Ce^2e^{-b_j/2} \le Ce^2e^{-x/2}$$

for $x \in [a_j, b_j]$. Letting F_1 denote the union of these type one intervals, we have

$$\int_{F_1} \tilde{h}(x)e^{-x}\,dx < \infty.$$

The second type of interval is one of the form $[a_j, b_j]$ where $b_j - a_j > 2$ and for which $\tilde{h}(x) \le e^{x/2}$ for $a_j \le x \le b_j$. If F_2 is the union of these

intervals, then

$$\int_{F_2} \tilde{h}(x)e^{-x}\,dx \le \int_0^\infty e^{-x/2}\,dx < \infty.$$

We are left with type three intervals $[a_j, b_j]$ where $-c_j \ge 2$ and there exists a point x_j between a_j and b_j with $\tilde{h}(x_j) \ge e^{x_j/2}$. If $x = \alpha a_j + (1-\alpha)b_j$, $0 \le \alpha \le 1$, then $x = c_j\alpha + b_j$. Hence

$$\int_{a_j}^{b_j} \tilde{h}(x)e^{-x}\,dx \le -Ce^{-b_j/2}c_j \int_0^1 \left(\alpha e^{-\alpha c_j}e^{c_j} + (1-\alpha)e^{-\alpha c_j}\right)\,d\alpha =$$

$$Ce^{-b_j/2}\frac{2-(e^{c_j}+e^{-c_j})}{c_j} \le -Ce^{-b_j/2}\frac{e^{c_j}+e^{-c_j}}{c_j}.$$

We enumerate these intervals consecutively. Recall that $c_j < -2$; hence $b_{j+1} > b_j + 2$, so $b_j > 2j$. Thus

$$-\sum_j e^{-b_j/2}\frac{e^{c_j}}{c_j} < \infty.$$

Since $\tilde{h}(a_j)e^{-a_j} \le Ce^{-b_j/2}$, $e^{b_j/2-a_j} \le C/\tilde{h}(a_j)$. Hence

$$-Ce^{-b_j/2}\frac{e^{-c_j}}{c_j} = C\frac{e^{b_j/2-a_j}}{b_j-a_j} \le C^2\frac{1}{\tilde{h}(a_j)(b_j-a_j)}.$$

But $\tilde{h}(a_{j+1}) \ge \tilde{h}(x_j) \ge e^{a_j/2}$. Since $a_{j+1} \ge 2j$, we see that

$$-\sum_j e^{-b_j/2}\frac{e^{-c_j}}{c_j} < \infty;$$

hence

$$\int_0^\infty \tilde{h}(x)e^{-x}\,dx < \infty.$$

$\square$

Final remark. For a C^∞ mapping $f = (f_1, \ldots, f_N) : S \to \mathbb{C}^N$ we can define $M_j(f) = \sup\{M_j(f_k) : 1 \le k \le N\}$. Referring to Section 2, we can define the associated function $\tau_f(r)$ and the sequence $\{t_n\}$ as well as Gevrey classes $G\{L_j\}$ of mappings. One can verify that the graph Γ_f of f in $\mathbb{C}^{N+1}$ is pluripolar if condition (7) is replaced by

$$\limsup_{n\to\infty} n^{1-1/(N+1)} \log t_n = \infty.$$

Moreover, an argument similar to one used in the proof of Theorem 1.2 can be used to prove that if $f \in G\{L_j\}$, where $L_j = o(j^{N+1})$, then Γ_f is pluripolar.

References

[BT] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), 1-40

[D] J. P. Demailly, *Mesures de Monge-Ampère et mesures plurisousharmoniques*, Math. Z. **194** (1987), 519-564.

[DF] K. Diederich and J. E. Fornæss, *A smooth curve in $\mathbb{C}^2$ which is not a pluripolar set*, Duke Math. J. 49 (1982), 931–936.

[Ka] Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover Publications, New York, 1976.

[K] M. Klimek, *Pluripotential Theory,* Oxford, Clarendon Press, 1991

[KP] S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser, 1992.

[L] P. Lelong, *Fonction de Green pluricomplexe et lemmes de Schwarz dans les espaces de Banach*, J. Math. Pures Appl. **68** (1989), 319-347.

[LMP] N. Levenberg, G. Martin, E.A. Poletsky, *Analytic disks and pluripolar sets,* Indiana Univ. Math. J. **41** (1992), 515-532.

[M] S. Mandelbrojt, *Sur les fonctions indéfinitement dérivables,* Acta Math., **72** (1940), 15-35.

[S] A. Sadullaev, *Plurisubharmonic Functions*, in *Several Complex Variables II*, Encyclopaedia of Mathematical Sciences Vol. 8, G. M. Khenkin and A. G. Vitushkin (Editors), Springer, 1994, 59-106.

[T] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, Macmillan, New York, 1963.

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